WS 2016/17

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# Symplectic Geometry

Homework 13

**Exercise 1.** (10 points)
Prove the following proposition discussed in class.
**Proposition:** Suppose a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ in a Hamiltonian way, with a moment map $\mu\colon M \to \mathfrak{g}^*$, and $H$ is a Lie subgroup of $G$. Let $j^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ denote the dual of the map $j\colon \mathfrak{h} = Lie(H) \to \mathfrak{g}$ induced by the inclusion of $H$ into $G$. Then the induced action of $H$ on $(M, \omega)$ is also Hamiltonian and $\mu_H := j^* \circ \mu\colon M \to \mathfrak{h}^*$ is a moment map.

**Exercise 2.** (10 points)
Fix integers $k_1, \dots, k_n$ and consider the $S^1$ action on $\mathbb{C}^n$ given by

$$e^{i\theta} * (z_1, \dots, z_n) = (e^{ik_1\theta} z_1, \dots, e^{ik_n\theta} z_n).$$

- Find a moment map $\mu$ for this circle action.
- Describe the level sets $\mu^{-1}(a)$ for $a \in Lie(S^1)^* \cong \mathbb{R}$.
- Consider $a$ such that $\mu^{-1}(a)$ is of maximal possible dimension (here: $2n-1$), and the induced $S^1$ action on $\mu^{-1}(a)$. Show that this action is free if and only if $|k_1| = \ldots = |k_n| = 1$.

The quotient, $\mu^{-1}(a)/S^1$, is called a weighted projective space and denoted by $\mathbb{CP}(k_1, \dots, k_n)$. The last statement implies that $\mathbb{CP}(k_1, \dots, k_n)$ is a smooth manifold if and only if $|k_1| = \ldots = |k_n| = 1$. Otherwise it is an orbifold.

**Exercise 3.** (10 points)
Recall the definition of the Poisson structure on $(M, \omega)$:

$$\{f, g\} := \omega(X_f, X_g).$$

Prove that $\{,\}$ satisfies the Jacobi identity.
Hint: You may want to use the following general formula for the derivative of a $k$ form $\eta$:

$$(d\eta)(X_1, \dots, X_{k+1}) =$$

$$\sum_{j=1}^{k+1} (-1)^{j+1} \mathcal{L}_{X_j}(\eta(X_1, \dots, \hat{X}_j, \dots, X_{k+1})) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}),$$

and the fact that for a vector field $X$ and a function $f$ we have that $\mathcal{L}_X(f) = df(X) = X(f)$.
As usually, I am using the symbol $\hat{X}_j$ to denote that $X_j$ is not one of the arguments.

**Exercise 4.** (10 points)
Suppose a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ in a Hamiltonian way, with a moment map $\mu\colon M \to \mathfrak{g}^*$. Take any $p \in M$ and let $G_p$ denote the stabilizer of $p$, and $\mathcal{O}_p$ the $G$ orbit through $p$, i.e.

$$G_p := Stab(p) = \{g \in G;\ g * p = p\}, \qquad \mathcal{O}_p := \{g * p;\ g \in G\}.$$

Show that

$$\ker(d\mu_{|p}) = (T_p\mathcal{O}_p)^{\omega_p}, \qquad \operatorname{im}(d\mu_{|p}) = \mathfrak{g}_p^0,$$

where $\mathfrak{g}_p^0 := \{\xi \in \mathfrak{g}^*;\ \forall_{X \in \mathfrak{g}_p}\ \langle \xi, X\rangle = 0\} \subset \mathfrak{g}^*$ denotes the annihilator of $\mathfrak{g}_p$.

Hand in: Thursday February 2nd
in the exercise session
in Übungsraum 1, MI